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Set-valued Skyline Fillings

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Abstract. Set-valued tableaux play an important role in combinatorial K-theory. Separately, semistandard skyline fillings are a combinatorial model for Demazure atoms and key polynomials. We unify these two concepts by defining a set-valued extension of semistandard skyline fillings and we then give analogues of results of J. Haglund, K. Luoto, S. Mason, and S. van Willigenburg. Additionally, we give a bijection between set-valued semistandard Young tableaux and C. Lenart's Schur expansion of the Grothendieck polynomial G_λ, using the uncrowding operator of V. Reiner, B. Tenner, and A. Yong.

Résumé. Les tableaux à valeurs sur des ensembles jouent un rôle important en K-théorie combinatoire. Séparément, remplissages des lignes d'horizon semi-standard sont un modèle combinatorie pour atomes de Demazure et ploynômes clés. Nous unifions ces deux concepts en définissant une extension des ensembles des emplissages des lignes d'horizon semi-standard et aisni donnons analogues des résultats de J. Haglund, K. Luoto, S. Mason, and S. van Willigenburg. En plus, donnon une bijection entre tableaux à valeurs sur des ensembles et l'expansion Schur du polynôme Grothendieck G_λ due á Lenart, en utilisant l'opérateur désertes de V. Reiner, B. Tenner, and A. Yong.

Keywords: Demazure atoms, skyline fillings, K-theory, quasisymmetric functions

1 Introduction

Textbook theory of the ring of symmetric functions concerns the Schur basis $\{s_\lambda\}$ and its combinatorial model of semistandard Young tableaux. In enumerative geometry, Schur functions are representatives for the Schubert classes in the cohomology ring of the Grassmannian. The symmetric Grothendieck function G_λ is an inhomogeneous deformation of s_λ and plays the analogous role in the K-theory of the Grassmannian [9]. A. Buch introduced set-valued tableaux as a combinatorial model for G_λ, thus providing a K-analogue of semistandard Young tableaux [1].

In representation theory, Schur functions are the characters of irreducible polynomial GL_n representations. Similarly, the key polynomials $\{\kappa_{\lambda,w}\}$ [10, 16] are the characters of Demazure modules of type A [2]. Moreover, for fixed λ, the key polynomials $\{\kappa_{\lambda,w}\}_{w\in S_n}$

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provide an interpolation between the single monomial x^λ and the Schur function s_λ. A. Lascoux and M.-P. Schützenberger introduced Demazure atoms to decompose the key polynomials [10], and thus Demazure atoms give a refinement of s_λ into nonsymmetric pieces [5, 13]. Combinatorially, S. Mason showed Demazure atoms are the generating function for semistandard skyline fillings [14].

The main goal of this paper is to unify these two extensions of Schur functions by defining semistandard set-valued skyline fillings. We then give generalizations of results about ordinary skyline fillings to show how our definition provides a K-analogue to Demazure atoms. This contributes to the study of K-analogues in the realm of algebraic combinatorics, see [7, 15, 17, 19] and the references therein.

1.1 Background

A *weak composition* (respectively *composition*) γ with k *parts* is a sequence of k nonnegative (respectively positive) integers $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$, and the *size* of γ is $|\gamma| = \sum_i \gamma_i$. The *skyline diagram* for γ with *basement* $\mathbf{b} = (b_1, \ldots, b_k)$ consists of k left-justified rows with γ_i boxes in row i, plus an additional column 0 containing the value b_i in row i. Furthermore, a *filling* is an assignment of positive integers to the boxes of the skyline diagram.

Skyline diagrams and fillings were introduced by J. Haglund, M. Haiman, and N. Loehr in their study of the nonsymmetric Macdonald polynomials [3]. *Triples* are central to the concept of a skyline filling and consist of three boxes on two rows $i < j$. As pictured, there are two types of triples depending on the relative lengths of the rows.

$$\begin{array}{|c|c|} \hline c & a \\ \hline \end{array} \qquad\qquad\qquad \begin{array}{|c|} \hline b \\ \hline \end{array}$$

$$\begin{array}{cc} & \vdots \end{array} \qquad\qquad\qquad\qquad \vdots$$

$$\begin{array}{c|c|} \cline{2-2} & b \\ \cline{2-2} \end{array} \qquad\qquad\qquad \begin{array}{|c|c|} \hline c & a \\ \hline \end{array}$$

Type A $\qquad\qquad$ Type B

$\gamma_i \geq \gamma_j$ $\qquad\qquad$ $\gamma_i < \gamma_j$

When the rows are weakly decreasing, a triple is an *inversion triple* if $b > c \geq a$ or $c \geq a > b$, and a *coinversion triple* when $a \leq b \leq c$. A filling is *semistandard* if

(M1) entries do not repeat in a column,

(M2) rows are weakly decreasing (including the basement), and

(M3) every triple (including those with basement boxes) is an inversion triple.

The notion of a *semistandard skyline filling* is due to S. Mason through her study of Demazure atoms [14]. Given a filling F, the *content* of F is the weak composition δ where

δ_i is the number of is in F, excluding any is in the basement. Then, the monomial x^F is $x^\delta = x_1^{\delta_1} x_2^{\delta_2} \ldots x_\ell^{\delta_\ell}$, and the *size* of F, denoted $|F|$, is $|\delta|$. Finally, the Demazure atom A_γ is

$$A_\gamma = \sum_F x^F$$

where the sum runs over all semistandard skyline fillings of γ with basement $b_i = i$ [14].

1.2 Definition of Set-Valued Skyline Fillings

We now extend the notion of semistandard fillings to set-valued fillings. A *set-valued filling* is an assignment of non-empty subsets of positive integers to the boxes of the skyline diagram. The maximum entry of each box is the *anchor entry* and all other entries are *free entries*. Let $\texttt{anchor}(r,c)$ (respectively $\texttt{free}(r,c)$) be the anchor entry (respectively set of free entries) of the box (r,c). A set-valued filling is *semistandard* if

(S1) entries do not repeat in a column,

(S2) rows are weakly decreasing where sets $A \geq B$ if $\min A \geq \max B$,

(S3) every triple of anchor entries is an inversion triple, and

(S4) if $a \in \texttt{free}(r,c)$ then for all $r' < r$, either $a > \texttt{anchor}(r',c)$ or $a < \texttt{anchor}(r',c+1)$.

Note that the concept of anchor entries is a key part of this definition, see Remark 2.5. By analogy with the Demazure case, we define combinatorial Lascoux atoms as the generating function for semistandard set-valued skyline fillings.

Definition 1.1. *For a weak composition γ, let $\mathsf{SetSkyFill}(\gamma)$ be the set of semistandard set-valued skyline diagrams of shape γ and basement $b_i = i$. Then the combinatorial Lascoux atom $\mathcal{L}_\gamma$ is*

$$\mathcal{L}_\gamma(x_1, \ldots, x_k; \beta) = \sum_{F \in \mathsf{SetSkyFill}(\gamma)} \beta^{|F| - |\gamma|} x^F.$$

Section 1.2 gives $\mathcal{L}_\gamma$ and the corresponding fillings for weak compositions that are rearrangements of $(2,1,0)$. Clearly, setting $\beta = 0$ yields A_γ, and thus $\mathcal{L}_\gamma$ is a inhomogeneous deformation of A_γ. This mostly shows combinatorial Lascoux atoms form a new (finite) basis of $\mathsf{Pol} = \mathbb{Z}[x_1, x_2, \ldots]$ – this is Proposition 2.2.

Definition 1.1 is our K-analogue of the Demazure atom. We will give generalizations of earlier results to support this view, and furthermore, the combinatorial Lascoux atom conjecturally satisfies the natural recurrence for K-theoretic Demazure atoms (see Conjecture 5.2).

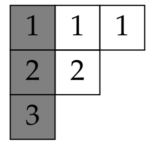
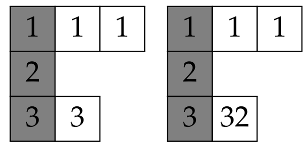
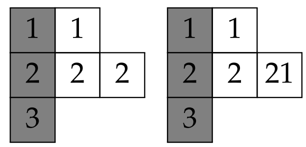
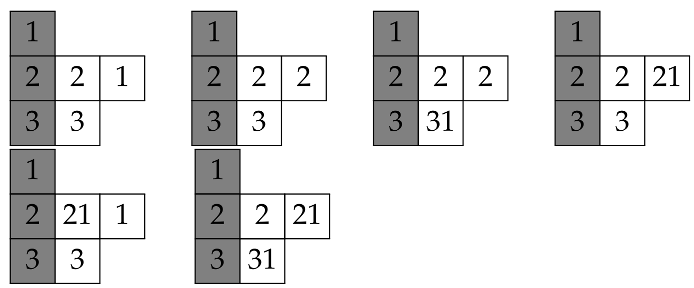
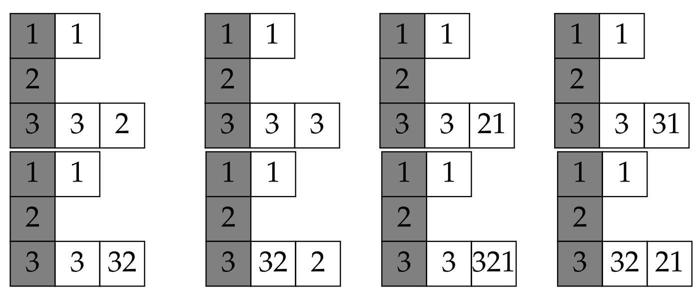
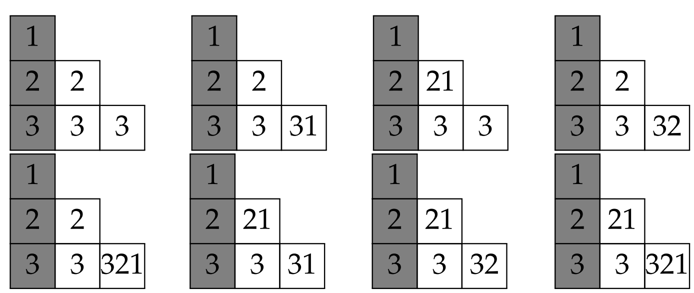
$\mathcal{L}_{210}$	$x_1^2x_2$	
$\mathcal{L}_{201}$	$x_1^2x_3 + \beta x_1^2x_2x_3$	
$\mathcal{L}_{120}$	$x_1x_2^2 + \beta x_1^2x_2^2$	
$\mathcal{L}_{021}$	$(x_1x_2x_3 + x_2^2x_3) + \beta(2x_1x_2^2x_3 + x_1^2x_2x_3) + \beta^2x_1^2x_2^2x_3$	
$\mathcal{L}_{102}$	$(x_1x_2x_3 + x_1x_3^2) + \beta(x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2x_3^2 + x_1x_2^2x_3) + \beta^2(x_1^2x_2x_3^2 + x_1^2x_2^2x_3)$	
$\mathcal{L}_{012}$	$x_2x_3^2 + \beta(2x_1x_2x_3^2 + x_2^2x_3^2) + \beta^2(x_1x_2^2x_3^2 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2) + \beta^3x_1^2x_2^2x_3^2$	

Figure 1: This example gives $\mathcal{L}_\gamma$ and the corresponding semistandard set-valued skyline fillings for weak compositions that are rearrangements of $(2, 1, 0)$.

1.3 Main Results

A *partition* is a weak composition such that the parts are weakly decreasing. For γ, define $\lambda(\gamma)$ as the unique partition with the same parts as γ. As $s_\lambda = \sum_{\lambda(\gamma)=\lambda} A_\gamma$, the Demazure atoms are a nonsymmetric refinement of the Schur functions [14]. We generalize this to G_λ and $\mathcal{L}_\gamma$, the K-analogues of s_λ and A_γ, respectively.

Theorem 1.2.

$$G_\lambda = \sum_{\lambda(\gamma)=\lambda} \mathcal{L}_\gamma.$$

QSym, the ring of quasisymmetric functions has an embedded copy of the ring of symmetric functions and itself embeds in the ring of formal power series. A function f is *quasisymmetric* if for any positive integers $\alpha_1, \ldots, \alpha_k$ and strictly increasing sequence of positive integers $i_1 < \ldots < i_k$,

$$[x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}]f = [x_1^{\alpha_1} \ldots x_k^{\alpha_k}]f.$$

J. Haglund, K. Luoto, S. Mason, and S. van Willigenburg also use Demazure atoms to define the *quasisymmetric Schur functions* $\{\mathcal{S}_\alpha\}$, which provide a quasisymmetric refinement of the Schur functions [4]. We generalize [4, Definition 5.1] to define the quasisymmetric Grothendieck functions.

Definition 1.3. *For a composition α, the quasisymmetric Grothendieck function $\mathcal{G}_\alpha$ is*

$$\mathcal{G}_\alpha = \sum_{\gamma^+=\alpha} \mathcal{L}_\gamma$$

where γ^+ is the composition formed by omitting parts of size 0 from γ.

By combining Theorem 1.2 and Definition 1.3, we decompose G_λ into quasisymmetric Grothendieck functions which generalizes the decomposition in [4, pg. 13].

Corollary 1.4.

$$G_\lambda = \sum_{\lambda(\alpha)=\lambda} \mathcal{G}_\alpha.$$

Theorem 1.5. *As α runs over all compositions, the functions $\{\mathcal{G}_\alpha\}$ form a basis for QSym.*

As seen below, the expansion of a power series f into Lascoux atoms allows us to determine if f is quasisymmetric or symmetric. If it is, the expansion allows us to determine if f is $\mathcal{G}_\alpha$- or G_λ-positive, which is often of interest, cf. [12, Section 1.1].

Proposition 1.6. *Suppose $f = \sum_\gamma c_\gamma \mathcal{L}_\gamma$. Then*

1. *f is quasisymmetric if and only if $c_\gamma = c_\delta$ for all $\gamma^+ = \delta^+$, and*
2. *f is symmetric if and only if $c_\gamma = c_\delta$ for all $\lambda(\gamma) = \lambda(\delta)$.*

Furthermore, if f is quasisymmetric (respectively symmetric), f is $\mathcal{G}_\alpha$-positive (respectively G_λ-positive) if and only if f is $\mathcal{L}_\gamma$-positive.

2 Combinatorial Lascoux Atoms

Let $\mathsf{Pol} = \mathbb{Z}[x_1, x_2, \ldots]$ and $\prec$ be the lexicographic order on monomials.

Lemma 2.1. *For $k = \max\gamma$,*

$$\mathcal{L}_\gamma = x^\gamma + \sum_{\substack{\delta \prec \gamma \\ \max\delta \le k}} c_{\gamma,\delta}\beta^{|\delta|-|\gamma|}x^\delta.$$

Proposition 2.2. *For all $f \in \mathsf{Pol}$, there is a unique expansion $f = \sum_\gamma c_\gamma \mathcal{L}_\gamma$, where all but finitely many $c_\gamma = 0$, i.e. $\{\mathcal{L}_\gamma\}$ forms a finite basis of* Pol.

We now define the bijections $\hat{\rho}$ and $\hat{\rho}^{-1}$ used to prove the G_λ expansion of Theorem 1.2. In the special case where there are no free entries, these are precisely the bijections ρ and ρ^{-1} given by Mason in [13].

A *set-valued reverse tableaux* is a filling of the shape λ with non-empty sets of positive integers with weakly (respectively strictly) decreasing rows (respectively columns). We use the convention that G_λ is the sum over set-valued reverse tableaux. Recall $\mathsf{SetSkyFill}(\gamma)$ is the collection of set-valued skyline fillings of shape γ and basement $b_i = i$ and let $\mathsf{SetRT}(\lambda)$ be the collection of set-valued reverse tableaux of shape λ. Then, we define the map

$$\hat{\rho} : \bigcup_{\lambda(\gamma)=\lambda} \mathsf{SetSkyFill}(\gamma) \to \mathsf{SetRT}(\lambda)$$

as follows. First, sort the anchor entries of each column into decreasing order and then place the free entries in the unique box in their column such that the columns remain strictly decreasing and the free entries remain free.

For the inverse $\hat{\rho}^{-1}$, start with an empty skyline diagram with basement $b_i = i$. Work by columns left to right, top to bottom and place each anchor entry in the first row such that weakly decreasing rows is preserved. When all anchor entries have been placed, place the free entries in the highest box in their column such that the rows are weakly decreasing and the free entries remain free.

Example 2.3. Given the filling $F = \begin{array}{|c|c|c|c|} \hline 1 & 1 & & \\ \hline 2 & & & \\ \hline 3 & 32 & 2 & 21 \\ \hline 4 & 4 & 431 & \\ \hline \end{array}$ we calculate $\hat{\rho}(F) = \begin{array}{|c|c|c|} \hline 4 & 43 & 21 \\ \hline 32 & 21 & \\ \hline 1 & & \\ \hline \end{array}$.

Theorem 2.4. *The map $\hat{\rho}$ is a bijection and $\hat{\rho}^{-1}$ is its inverse.*

Remark 2.5. One might expect a semistandard set-valued skyline filling to be a filling such that any selection of one number from each box is a semistandard skyline filling. However, then the left tableau below would not be semistandard as the right tableau violates the triple condition in rows 2 and 3. Compare this with [6, Section 1.2].

1	1	1
2		
3	32	

1	1	1
2		
3	2	

3 Quasisymmetric Grothendieck Functions

Recall that a function f is quasisymmetric if for any positive integers $\alpha_1, \ldots, \alpha_k$ and strictly increasing sequence of positive integers $i_1 < i_2 < \ldots < i_k$, $[x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}]f = [x_1^{\alpha_1} \ldots x_k^{\alpha_k}]f$. Equivalently f is quasisymmetric if and only if for all i, f is invariant under switching x_i and x_{i+1} *except not in monomials that have both*. Thus when f is modeled by a combinatorial object, we expect f to be quasisymmetric if and only if the combinatorial object is governed by rules depending *only* on relative order.

Therefore Demazure and Lascoux atoms are not quasisymmetric because the basement $b_i = i$ forces the anchor entry at position $(i, 1)$, if it exists, to be i. In [4], the quasisymmetric Schur function was originally defined $\mathcal{S}_\alpha = \sum_{\gamma^+=\alpha} A_\gamma$, where γ^+ is the composition formed from γ by omitting parts of size 0. $\mathcal{S}_\alpha$ was then shown to be the sum over *semistandard composition tableaux*, skyline fillings of a composition α with no basement and strictly increasing entries from top to bottom along the first column.

Thus we define *semistandard set-valued composition tableaux* as fillings of a composition shape α with non-empty subsets of positive integers such that

(Q1) entries weakly decrease along rows,

(Q2) anchor entries form a semistandard composition tableau, and

(Q3) if $a \in \mathtt{free}(r, c)$ then for all $r' < r$, either $a > \mathtt{anchor}(r', c)$ or $a < \mathtt{anchor}(r', c+1)$.

Let $\mathsf{SetCompTab}(\alpha)$ be the collection of semistandard set-valued composition tableaux of shape α. Inserting rows of size 0 allows the anchor entries of the first column to be any increasing sequence, and thus

$$\mathcal{G}_\alpha := \sum_{\gamma^+=\alpha} \mathcal{L}_\gamma = \sum_{T \in \mathsf{SetCompTab}(\alpha)} \beta^{|T|-|\alpha|} x^T.$$

Since all the rules governing a semistandard set-valued composition tableau only depend on the relative order of the entries in each box, we expect $\mathcal{G}_\alpha$ to be quasisymmetric. The following results are used to prove Theorem 1.5 from the introduction.

Proposition 3.1. *The function $\mathcal{G}_\alpha$ is quasisymmetric.*

Proposition 3.2. *For compositions α with at most n parts, $\{\mathcal{G}_\alpha(x_1, \ldots, x_n, 0, \ldots)\}$ forms a basis of $QSym_n$, the ring of quasisymmetric polynomials in n variables.*

4 Schur Expansion of G_λ

We now provide a new link between ordinary and set-valued tableaux by giving a bijection between the combinatorial objects in C. Lenart's Schur expansion of G_λ and semistandard set-valued tableaux. This section is independent from the remainder of the paper. Let

$$\mathsf{S}(\lambda) = \{F : F \text{ is a semistandard set-valued tableaux of shape } \lambda\}$$

and

$$\mathsf{L}(\lambda) = \left\{ (T,U) : \begin{array}{c} T \text{ is row and column strict of shape } \mu/\lambda \\ \text{with entries of row } i \text{ between 1 and } i-1, \text{ and} \\ U \text{ is semistandard of shape } \mu \end{array} \right\}.$$

Then we have the following two descriptions of G_λ due to C. Lenart and A. Buch, respectively.

Theorem 4.1 ([11], Theorem 2.2)**.**

$$G_\lambda = \sum_{\lambda \subseteq \mu} \beta^{|\mu|-|\lambda|} g_{\lambda,\mu} s_\mu = \sum_{(T,U) \in \mathsf{L}(\lambda)} \beta^{|T|} x^U$$

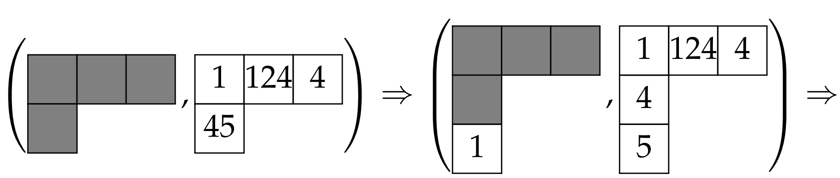
where $g_{\lambda,\mu} = \#\{T : (T,U_0) \in \mathsf{L}(\lambda)\}$ for U_0, any fixed semistandard tableaux of shape μ.

Theorem 4.2 ([1], Theorem 3.1)**.**

$$G_\lambda = \sum_{F \in \mathsf{S}(\lambda)} \beta^{|F|-|\lambda|} x^F.$$

We give a bijection $\texttt{uncrowd} : \mathsf{L}(\lambda) \to \mathsf{S}(\lambda)$ using a repeated application of the recent uncrowding operation of V. Reiner, B. Tenner, and A. Yong [17]. Given a set-valued tableaux F of shape λ, begin with $T = \lambda/\lambda$ and $U = F$. Read the boxes of U from bottom to top, right to left. While the current box has more than one number, *uncrowd* the box by iteratively removing the largest number from the box and RSK-inserting into the row below. During each step, a box will be added to U, and in the corresponding box of T record $k - i$ where k is the row of the new box and i is the original row of the number inserted.

Example 4.3. Let $F = \begin{array}{|c|c|c|} \hline 1 & 124 & 4 \\ \hline 45 \\ \cline{1-1} \end{array}$. Then $\texttt{uncrowd}(F)$ is calculated as follows:



$$\left(\begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & 1 & \\ \cline{1-2} 1 & & \\ \cline{1-1} \end{array} , \begin{array}{|c|c|c|} \hline 1 & 12 & 4 \\ \hline 4 & 4 & \\ \cline{1-2} 5 & & \\ \cline{1-1} \end{array} \right) \Rightarrow \left(\begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & 1 & \\ \cline{1-2} 1 & & \\ \cline{1-1} 3 & & \\ \cline{1-1} \end{array} , \begin{array}{|c|c|c|} \hline 1 & 1 & 4 \\ \hline 2 & 4 & \\ \cline{1-2} 4 & & \\ \cline{1-1} 5 & & \\ \cline{1-1} \end{array} \right) = \texttt{uncrowd}(F)$$

Theorem 4.4. *The map* `uncrowd` *is a bijection from* $\mathsf{S}(\lambda)$ *to* $\mathsf{L}(\lambda)$.

In the proof of this theorem, we construct the inverse $\texttt{crowd} : \mathsf{L}(\lambda) \to \mathsf{S}(\lambda)$ again by iterating the process of V. Reiner, B. Tenner, and A. Yong. Given a pair (T, U), consider the tableaux $\tilde{T}$ formed by replacing each x in row k of T with $k - x$. Let i be the minimal value of $\tilde{T}$ and choose the lowest inner corner containing i. Reverse-RSK from this corner in U, but stop after reverse-bumping a value, say x, out of row $i + 1$. Instead of bumping a value out of row i, add x to the unique box b of row i such that x is free and row i is weakly decreasing. Then remove the now empty corner from U and the corresponding box from $\tilde{T}$. Repeat this process until $\tilde{T}$ is empty.

5 Conjectures

We have defined the Lascoux atoms combinatorially in terms of set-valued skyline fillings, but there is also a natural definition based on isobaric divided difference operators. Let s_i act on polynomials by switching x_i and x_{i+1}. Then, we have the operators

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}} \qquad \pi_i = \partial_i x_i \qquad \hat{\pi}_i = \pi_i - 1.$$

These operators all satisfy the braid relations and so given $w \in S_n$, we define ∂_w by $\partial_w = \partial_{a_1} \ldots \partial_{a_k}$ (and π_w and $\hat{\pi}_w$ analogously) where $a_1 \ldots a_k$ is any reduced word of w. Given a weak composition γ, let $w(\gamma)$ be the shortest permutation that sends $\lambda(\gamma)$ to γ. For example, for $\gamma = 1021$, $\lambda(\gamma) = 211$ and $w(\gamma) = 3142$.

The Demazure character is $\kappa_\gamma = \pi_{w(\gamma)} x^{\lambda(\gamma)}$ and the Demazure atom is $A_\gamma = \hat{\pi}_{w(\gamma)} x^{\lambda(\gamma)}$ where $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \ldots$. In [8], Lascoux defined K-theoretic deformations of the Demazure characters using modified operators that still satisfy the braid relations:

$$\tilde{\partial}_i = \partial_i (1 + \beta x_{i+1}) \qquad \tau_i = \pi_i (1 + \beta x_{i+1}) \qquad \hat{\tau}_i = \tau_i - 1.$$

Then the Lascoux polynomial is $\Omega_\gamma = \tau_{w(\gamma)} x^{\lambda(\gamma)}$ and the Lascoux atom is $\hat{\mathcal{L}}_\gamma = \hat{\tau}_{w(\gamma)} x^{\lambda(\gamma)}$. By manipulating the operators above, we obtain the following decomposition of the Lascoux polynomial into Lascoux atoms that matches the Demazure case.

Theorem 5.1.

$$\Omega_\delta = \sum_{\gamma \leq \delta} \hat{\mathcal{L}}_\gamma$$

where $\gamma \leq \delta$ *if* $\lambda(\gamma) = \lambda(\delta)$ *and* $w(\gamma) \leq w(\delta)$ *in (strong) Bruhat order.*

A conjectural combinatorial model for Ω_γ using K-Kohnert diagrams was given by C. Ross and A. Yong in [18], but there are no proven combinatorial rules for Ω_γ or $\hat{\mathcal{L}}_\gamma$. However, we have checked the following conjectures for all weak compositions γ with at most 8 boxes and at most 8 rows, both which generalize the Demazure case.

Conjecture 5.2.

$$\hat{\mathcal{L}}_\gamma = \mathcal{L}_\gamma = \sum_{F \in \mathsf{SetSkyFill}(\gamma)} \beta^{|F|-|\gamma|} x^F.$$

Conjecture 5.3.

$$\Omega_\gamma = \sum_F \beta^{|F|-|\gamma|} x^F$$

where the sum runs over all semistandard set-valued skyline fillings of shape γ^* (the parts of γ written in reverse order) and basement $b_i = n - i + 1$.

In [5], J. Haglund, K. Luoto, S. Mason, and S. van Willigenburg refine the Littlewood-Richardson rule to give the expansion of $A_\gamma \cdot s_\lambda$ into Demazure atoms. O. Pechenik and A. Yong [15] develop the theory of genomic tableaux to describe multiplication in K-theory. We conjecture the natural genomic analogue of the rule of J. Haglund et. al. extends to Lascoux atoms.

When δ, γ are weak compositions with $\gamma_i \le \delta_i$ for all i, a *skew skyline diagram* of shape δ/γ is formed by taking the skyline diagram of shape δ and given basement and extending the basement into the cells of γ. If n is the largest entry allowed in the filling, a *large basement* is such all basement entries of the basement are larger than n and decrease from top to bottom. As seen in [5], with a large basement, the exact basement entries do not determine valid skyline fillings and thus we denote it by '$*$'.

A *genomic* filling is a filling of δ/γ with labels i_j where i is a positive integer and for each i, $\{j | i_j \text{ appears in the filling}\} = \{1, \ldots, k_i\}$ for some nonnegative integer k_i. The set of labels $\{i_j\}$ for all j is the *family* i, while the set of all labels i_j for fixed i and j is the *gene* i_j. The *content* of a genomic filling is $(k_1, k_2, \ldots)$. The *column reading word* of a skyline filling reads the entries of the boxes (excluding the basement) in columns from top to bottom, right to left. A genomic filling is *semistandard* if

(G1) at most one entry from a family (respectively gene) appears in a column (respectively row),

(G2) the label families are weakly decreasing along rows,

(G3) every triple with three distinct genes is an inversion triple comparing families, and

(G4) for every i, the genes appear in weakly decreasing order along the reading word.

A word is *reverse lattice* if at any point and any i we have always read more $i+1$s than is. A genomic filling is *reverse lattice* if for any selection of exactly one label per gene, the column reading word is reverse lattice.

Conjecture 5.4.
$$\mathcal{L}_\gamma \cdot G_\lambda = \sum_\delta \tilde{a}^\delta_{\gamma,\lambda} \mathcal{L}_\delta$$
where $\tilde{a}^\delta_{\gamma,\lambda}$ is the number of reverse lattice, genomic semistandard skyline fillings of skew-shape δ/γ (using a large basement) with content λ^*.

Example 5.5. $\tilde{a}^{314}_{102,21} = 2$ and the two witnessing fillings are

$$\begin{array}{cccc} * & 2_1 & 1_1 & \\ 1_1 & & & \\ * & * & 2_1 & 2_2 \end{array} \qquad\qquad \begin{array}{cccc} * & 2_1 & 1_1 & \\ 2_1 & & & \\ * & * & 2_1 & 2_2 \end{array} .$$

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